Theorems on Trees (Contd.)

Thm Every connected graph has at least one spanning tree.

Pf:- Let G be a connected graph. If G has no circuit then it is its own spanning tree. If G has a circuit then delete an edge from the circuit. The graph obtained by removing an edge from a circuit in G will remain connected. If there are more circuits then repeat this process until we get a connected, circuit less graph that contains all the vertices of graph G. This graph will then be a spanning tree of graph G.

(Proved)

Thus with respect to any of its spanning trees, a connected graph with n vertices and e edges has $(n-1)$ tree branches and $(e-n+1)$ chords.

Pf → Let G be a graph with n vertices and e edges. Let T be any spanning tree of G. Then T has n vertices and $(n-1)$ edges. The remaining edge

$e-(n-1) = e-n+1$

will be chords of G.

Thus the graph G has $(n-1)$ tree branches and $(e-n+1)$ chords.

(Proved)

Theorem on Cut-Sets

Thm: Every cut-set in a connected graph G contains at least one branch of every spanning tree of G.

Pf→ Let G be a connected graph and let T be any spanning tree of G. Let S be an arbitrary cut-set in G. If possible, suppose S does not have any edge of T. Then removal of the set S from G will not disconnect the graph because the graph G−S contains T which is connected subgraph of G. Which is a contradiction of the definition of cut-set. Hence S must have at least one edge of T.

(Proved)

Thm: Any minimal set of edges containing at least one branch of every spanning tree of a connected graph G is a cut-set.

Pf→ Let G be a connected graph and let S be a minimal set of edges containing at least one branch of every spanning tree of G. Let us consider G−S,

which is a subgraph of G after removal of S from G. Since S contains at least one branch of every spanning tree of G, therefore G-S contains no spanning tree of G. Hence G-S is disconnected.
Also, the set S is a minimal set of edges such that G-S is disconnected. Hence S is a cut-set.
Therefore any minimal set of edges which contains at least one branch of every spanning tree of a connected graph G is always a cut-set.